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Condensation and Magnetization of the Relativistic Bose Gas

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Abstract

We present a simple proof of the absence of Bose–Einstein condensation of a relativistic boson gas, in any finite local magnetic field in less than five dimensions. We show that the relativistic charged boson gas exhibit a genuine Meissner–Ochsenfeld effect of the Schafroth form at fixed supercritical density. As in the well–known non–relativistic case, this total expulsion of a magnetic field is caused by the condensation of the Bose gas at vanishing magnetic field. The result is discussed in the context of kaon condensation in neutron stars.

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In this letter we study some aspects of an ideal charged Bose gas at finite temperature T and chemical potential μ in presence of a static uniform magnetic field. Magnetic fields B associated with compact astrophysical objects may range between $B = \mathcal{O}(10^4)$ T for magnetic white dwarfs to $B = \mathcal{O}(10^{10})$ T for supernovae [1]. As a reference we recall that the characteristic magnetic field in QED is $m_e^2/e = \mathcal{O}(10^9)$ T. For such systems one may expect that also thermal effects are of importance. In the absence of a magnetic field it is known that the relativistic charged boson gas exhibits a Bose–Einstein condensation (see e.g. Ref.[2]). In the presence of a magnetic field the non–relativistic charged boson gas was studied by Schafroth [3]. The relativistic system has recently been extensively considered by Daicic et al. [4]. However, we do not agree with their conclusions about the relativistic Meissner–Ochsenfeld effect.

According to Ref.[5], introducing an external field is equivalent to introducing an external current independent of the dynamics of the system considered. Including the term $\mathcal{L}_{\text{ext}} = j^{\nu}_{\text{ext}} A_\nu$ in the classical Lagrangian for scalar QED, Euler–Lagrange's equation of motion for A_ν reads

$$\partial^\mu F_{\mu\nu} = j_\nu + j_\nu^{\text{ext}} \quad , \tag{1}$$

where j_ν is the induced current. When quantizing the particles but not the gauge field we may, in the uniform magnetostatic case, perform the functional integral over the scalar field, and write the effective Lagrangian density as

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_{\text{eff}}^{\text{vac}} + \mathcal{L}_{\text{eff}}^{\beta,\mu} + \mathcal{L}_{\text{ext}} \quad , \tag{2}$$

where $\mathcal{L}_0 = -\frac{1}{2}B^2$ is the tree level term, $\mathcal{L}_{\text{eff}}^{\text{vac}}$ is the one–loop vacuum correction, and $\mathcal{L}_{\text{eff}}^{\beta,\mu}$ is the thermal contribution. In terms of the (average microscopic) magnetic induction $\mathbf{B} = \nabla \times \mathbf{A}$, and the external magnetic field $\mathbf{H}$, such that $\nabla \times \mathbf{H} = \mathbf{j}_{\text{ext}}$, we may (upon neglecting a surface term) write $\mathcal{L}_{\text{ext}} = \mathbf{B} \cdot \mathbf{H}$. The mean-field equation then follows from a minimization of the effective action (in natural units $\alpha = e^2/4\pi$)

$$\mathbf{B} = \mathbf{H} + \mathbf{M}(\mathbf{B}) \quad , \tag{3}$$

where the average microscopic magnetization is defined by

$$M_i(x) = \frac{\partial}{\partial B_i(x)}(\mathcal{L}_{\text{eff}}^{\text{vac}} + \mathcal{L}_{\text{eff}}^{\beta,\mu}) \quad , \tag{4}$$

such that the expectation value of the induced current is $\langle \mathbf{j} \rangle = \nabla \times \mathbf{M}$. Considering the external field as the acting field (i.e. the field felt by the particles) as in Ref.[4], means

that the induced current is neglected in Eq.(1), and lead to erroneous conclusions about the relativistic Meissner effect. In the case of Meissner effect, the magnetization appears to be stronger than the external field, as explained below, and thus certainly may not be neglected. In the original work by Schafroth [3] and also in Ref.[6], B was used as the acting field, consistent with Eqs.(3,4). The contribution from the vacuum polarization $\mathcal{L}_{\rm eff}^{\rm vac}$ to the magnetization is negligible for small magnetic fields, and of no importance when considering the topics discussed here, so it will be neglected in what follows.

In Ref.[7] the thermal part of the effective Lagrangian $\mathcal{L}_{\rm eff}^{\beta,\mu}$ for the relativistic charged boson gas in a homogeneous magnetic field, neglecting all boundary effects, was calculated and found to be related to the free energy F, and grand partition function $Z(B,T,\mu)$, according to $\mathcal{L}_{\rm eff}^{\beta,\mu} = \log Z/V\beta = -F/V$. Generalizing to $d+1$ dimensional spacetime [6], the free energy density $f_d = F_d/V_d$ is

$$f_d = -\frac{eB}{2^{d-1}\pi^{d/2}\Gamma(\frac{d}{2})}\sum_{n=0}^{\infty}\int_0^{\infty} dp\frac{p^{d-1}}{E_n(p)}\Big(f_B^+(E_n(p)) + f_B^-(E_n(p))\Big) \quad , \tag{5}$$

where $f_B^\pm$ are the one-particle distributions $f_B^\pm(\omega) = (e^{\beta(\omega\mp\mu)}-1)^{-1}$, and the energy is given by

$$E_n(p) = \sqrt{m^2+p^2+(2n+1)eB} \quad . \tag{6}$$

Following the steps in Ref.[7], separating the particle(+) and antiparticle(−) contributions, and introducing dimensionless quantities ($\bar\beta \equiv m\beta,\ \bar\mu \equiv \mu/m,\ \bar f_d \equiv f_d/m^{d+1},\ \bar\rho_d \equiv \rho_d/m^d,\ \bar B \equiv eB/m^2,\ \bar H \equiv eH/m^2$), we rewrite Eq.(5) as

$$\bar f_d^\pm = -(\frac{1}{4\pi})^{\frac{d+1}{2}}\sum_{k=1}^{\infty}\int_0^{\infty}\frac{ds}{s}s^{-(\frac{d+1}{2})}\exp(-s-\frac{\bar\beta^2k^2}{4s})\frac{\bar Bs}{\sinh\bar Bs}e^{\pm k\bar\beta\bar\mu} \quad . \tag{7}$$

The integral and sum here are absolutely convergent, but if we expand in powers of $\bar B$, the series is only asymptotic, and as $\bar\mu$ assumes its critical value at the lowest energy level $\bar\mu_c = E_0(0)/m = \sqrt{1+\bar B}$, the coefficients become divergent. However, we may still in a simple manner discuss the analytical behaviour of the free energy and the magnetization, using the following inequality

$$(1+2\bar Bs)e^{-\bar Bs} > \frac{\bar Bs}{\sinh\bar Bs} > (1+\bar Bs)e^{-\bar Bs} \quad , \tag{8}$$

which we write as

$$\frac{\bar Bs}{\sinh\bar Bs} \lesseqgtr (1+c_0\bar Bs)e^{-\bar Bs}, \quad c_0\in[1,2] \quad , \tag{9}$$

and similarly for the purpose of the magnetization

$$\frac{\partial}{\partial \bar{B}} \frac{\bar{B}s}{\sinh \bar{B}s} \lessgtr -c_1 \bar{B} s^2 e^{-\bar{B}s}, \qquad c_1 \in [\frac{1}{3}, 2] \quad . \tag{10}$$

Introducing the function ($x = \bar{\beta} k s$)

$$\gamma^{\pm}(x) \equiv \bar{\beta}[\frac{\bar{\mu}_c^2}{x}(x - x_0)^2 + (\bar{\mu}_c \mp \bar{\mu})] \quad , \; x_0 \equiv 1/(2\bar{\mu}_c) \quad , \tag{11}$$

and identifying a Jonquière's function with an exponential argument $\psi_a(z) \equiv \sum_{k=1}^{\infty} e^{-kz}/k^a$, the charge density $e\bar{\rho} = e(\bar{\rho}^+ + \bar{\rho}^-) = -e\frac{\partial \bar{f}}{\partial \bar{\mu}}$, and the magnetization $\bar{M}_d = -4\pi\alpha \frac{\partial \bar{f}_d}{\partial \bar{B}}$ are written

$$\bar{\rho}_d^{\pm} \quad \lessgtr \quad \pm(\frac{1}{4\pi\bar{\beta}})^{\frac{d+1}{2}} \bar{\beta} \int_0^{\infty} \frac{dx}{x} x^{-(\frac{d+1}{2})} \left\{ \psi_{\frac{d-1}{2}}[\gamma^{\pm}(x)] + c_0 \bar{B} \bar{\beta} x \psi_{\frac{d-3}{2}}[\gamma^{\pm}(x)] \right\} \quad , \tag{12}$$

$$\bar{M}_d^{\pm} \quad \lessgtr \quad -4\pi\alpha\, c_1 \bar{B} (\frac{1}{4\pi\bar{\beta}})^{\frac{d+1}{2}} \bar{\beta}^2 \int_0^{\infty} \frac{dx}{x} x^{-(\frac{d+1}{2})} x^2 \psi_{\frac{d-3}{2}}[\gamma^{\pm}(x)] \quad . \tag{13}$$

The charge density naturally splits into two parts $\bar{\rho} \equiv \bar{\rho}_{\rm reg} + \bar{\rho}_{\rm div}$, where $\bar{\rho}_{\rm reg}$, the first term in Eq.(12), for a vanishing magnetic field is the charge density of noncondensed states, and $\bar{\rho}_{\rm div}$ is the second term in Eq.(12). We shall now investigate $\bar{\rho}_{\rm div}$ for $\bar{\mu} \to \bar{\mu}_c$, in order to see if the magnetized Bose gas can form a condensate. The leading behaviour of $\psi_{\frac{d-3}{2}}(\gamma^+)$ close to $x = x_0$ is for $\frac{d-3}{2} \le 1$, or d even [3]

$$\psi_{\frac{d-3}{2}}[\gamma^{\pm}(x)] \sim \Gamma(\frac{5-d}{2}) \left\{ \bar{\beta}[\frac{\bar{\mu}_c^2}{x}(x - x_0)^2 + (\bar{\mu}_c \mp \bar{\mu})] \right\}^{\frac{d-5}{2}} \quad . \tag{14}$$

For positive chemical potential ($\bar{\mu} < 0$ can be treated similarly) we thus see that $\bar{\rho}_{d,\rm div}^-$ remains finite, whereas at $\bar{\mu} = \bar{\mu}_c$, $\psi_{\frac{d-3}{2}}[\gamma^+(x)] \sim (x - x_0)^{d-5}$ so that $\bar{\rho}_{d,\rm div}^+$ diverges for $d \le 4$. Actually, also $\bar{\rho}_{d,\rm reg}^+$ diverges for $d \le 2$. At finite magnetic field, for an arbitrary density $\bar{\rho}_{d \le 4}$ there is thus a value of the chemical potential $\bar{\mu}$ for any inverse temperature β, such that $\bar{\rho}_d = \bar{\rho}_{d,\rm reg} + \bar{\rho}_{d,\rm div}$, hence the magnetized Bose gas does not condense for $d \le 4$ (and in the absence of the field the gas does not condense for $d \le 2$). Actually this can be seen in a physically more illuminating way. The divergent contribution to the charge density should come from the lowest Landau level ($n = 0$). Separating out that level we have according to Eq.(5), after a change of variables of summation and integration

$$\begin{aligned} \bar{\rho}_d^+ \quad = \quad & -\frac{\bar{B}}{2^{d-2}\pi^{d/2}\Gamma(\frac{d}{2}-1)} \int_0^{\infty} dx x^{d-3} \frac{1}{\exp[\bar{\beta}(\sqrt{1 + x^2 + \bar{B}} - \bar{\mu})] - 1} \\ & -\frac{\bar{B}}{2^{d-2}\pi^{d/2}\Gamma(\frac{d}{2}-1)} \sum_{n=0}^{\infty} \int_0^{\infty} dx x^{d-3} \frac{1}{\exp[\bar{\beta}(\sqrt{1 + x^2 + (2n+3)\bar{B}} - \bar{\mu})] - 1} . \end{aligned} \tag{15}$$

The first term is easily seen to diverge at $\bar{\mu} = \bar{\mu}_c$ for $d \leq 4$, and the second term is exactly of the same form as if the lowest Landau level was included, but with the lowest energy $(\sqrt{1+3\bar{B}})$ always larger than the chemical potential $(\bar{\mu} \leq \bar{\mu}_c)$, for finite B. It thus follows from the above inequalities that this sum always is finite. Hence even though no true Bose–Einstein condensate can form for $d \leq 4$, the lowest Landau level can play the role of the groundstate and accommodate a large charge density.

Let us now consider the physically most relevant case of $d = 3$, suppress the dimensional index and return to dimensionful quantities. In the case of vanishing magnetic field $\rho_{\rm div} \equiv 0$, and $\rho_{\rm reg}$ is finite as $\mu \to \mu_c(B=0) = m$. At fixed temperature there is thus a critical density

$$\rho_c(T) = \rho_{\rm reg}(\mu = m, T, B = 0) = \frac{Tm^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k} K_2(k\beta m) \sinh(k\beta m) \quad . \tag{16}$$

Condensation occurs if this critical density is exceeded. In the limit $\mu \to \mu_c$ the integrals in Eqs.(12, 13) are dominated by x close to x_0, and we obtain the leading behaviour $(\mu_c = \sqrt{m^2 + eB})$

$$\rho^+_{\rm div} \quad \lessgtr \quad \frac{1}{4\pi\sqrt{2}} c_0 eB\, T \sqrt{\frac{\mu_c}{\mu_c - \mu}} \quad , \tag{17}$$

$$M \quad \lessgtr \quad -\frac{e}{8\pi\sqrt{2}} c_1 eB \frac{T}{\mu_c} \sqrt{\frac{\mu_c}{\mu_c - \mu}} \quad . \tag{18}$$

We now whish to consider the magnetization in the limit of vanishing magnetic field $B \to 0$, at fixed supercritical density $\rho > \rho_c$, i.e when a condensate is formed. Then we must have that $\bar{\mu} \to \bar{\mu}_c$, and in this limit $\bar{\rho}_{\rm reg} \to \bar{\rho}_c$, so that Eqs.(17, 18) give that the magnetization is approaching a constant. The obtained magnetization law

$$M(B \to 0) = -\frac{e}{2m} \frac{c_1}{c_0} (\rho - \rho_c) \quad , \tag{19}$$

is exactly of the Schafroth form [3], who derived it in the non–relativistic case with the constant $c_1/c_0 = 1$. We may actually determine the limiting values of c_0 and c_1 here, using the previous method of separating out the lowest Landau level. Taking the limit $B \to 0$ for all higher Landau levels we find the supercritical density

$$\begin{aligned} \rho \quad &= \quad \frac{eB}{2\pi^2} \int_0^\infty dp \frac{1}{\exp[\beta(E_0(p) - \mu)] - 1} \\ &+ \frac{1}{2\pi^2} \int_0^\infty dp\, p^2 \left(f_B^+(\sqrt{p^2 + m^2}) + f_B^-(\sqrt{p^2 + m^2}) \right) \Big|_{\mu = m} \quad , \end{aligned} \tag{20}$$

and the magnetization

$$M \;=\; -\frac{e^2B}{4\pi^2}\int_0^\infty \frac{dp}{E_0(p)}\frac{1}{\exp[\beta(E_0(p)-\mu)]-1}\;. \tag{21}$$

In the $n=0$ term we let μ approach $\sqrt{m^2+eB}$ in such a way that a prescribed ρ is obtained. We thus find $c_0=c_1=2$, in agreement with Shafroth's [3] non–relativistic result. In the ultra–relativistic ($T\gg m$) and non–relativistic ($T\ll m$) limit, Eq.(16) gives the well–known result

$$\rho_c \approx \begin{cases} \frac{1}{3}T^2m & ,\; T\gg m \\ \zeta(\frac{3}{2})(\frac{1}{2\pi})^{\frac{3}{2}}(Tm)^{3/2} & ,\; T\ll m \end{cases} \;. \tag{22}$$

Below the critical temperature we can therefore write the magnetization

$$M(B\to 0) \approx -\frac{e}{2m}\rho \begin{cases} [1-(T/T_c)^2] & ,\; T\gg m \\ [1-(T/T_c)^{3/2}] & ,\; T\ll m \end{cases} \;. \tag{23}$$

where T_c is defined by the condition that $\rho_c(T=T_c)=\rho$. There is thus a critical field $H_c\equiv -M(B\to 0)$, such that for external fields smaller than H_c Eq.(3) has no solution, and the field is expelled from the Bose gas. This is the well–known Meissner–Ochsenfeld effect. The expulsion of the field is caused by surface currents, and since we do not consider surface effects here, they appear as external currents in our formalism. The origin of the Meissner–Ochsenfeld effect is here, as well as in the non-relativistic case, that the total free energy is minimized if a condensate is formed, which is only possible in vanishing magnetic field, and is not connected to the high temperature pair production as claimed in Ref.[4]. The perfect expulsion of the externally applied field increases the free energy per unit volume by [8] $H^2_{\rm appl}/2$, which causes penetration at supercritical field strengths. If we instead keep $\mu=m$ fixed as $B\to 0$, Eq.(18) gives ($\mu_c\simeq m+eB/2m$)

$$M \simeq -e\frac{c_1}{8\pi}\sqrt{eB}\,T\;. \tag{24}$$

The square root magnetization law in Eq.(24) was obtained in Ref.[4][1], where the corresponding $c_1=6\sqrt{2}\pi\,\zeta[-1/2,1/2]\simeq 1.6$ was found. Considering the externally applied field as the acting field made the authors of Ref.[4] draw erroneous conclusions about the Meissner effect from Eq.(24). We claim that the magnetization law of Eq.(19), leading to

[1] Here an expansion of the form $\mu=m+g(\beta,\beta_c,B)$, for $g\ll eB/m$ was used. In the limit $B\to 0$ this is only valid for $g\equiv 0$, and thus this corresponds to keeping the chemical potential (and not the density as was intended) fixed for small magnetic fields.

genuine Meissner–Ochsenfeld effect, may be derived from the expressions in Ref.[4] if a supercritical density is correctly held fixed.

There has recently been some discussion about condensation of K^- mesons in the core of neutron stars and its influence on the equation of state [9]. Since there are also very large magnetic fields in some neutron stars it is interesting to ask whether the field can influence the condensation. If magnetic flux is trapped in the inner core when the protons become superconducting it is believed that flux tubes are formed with field strengths much higher than on the surface. The details of the formation and dynamics of such flux tubes are not known so we shall only estimate some relevant quantities related to the kaons. We can here safely put $T = 0$ compared to the kaon effective mass $m_K^* \simeq 210$ MeV and use Eq.(23) to compute the critical field strength. For the typical value $\rho_K \simeq 0.1$ fm^{-3} we find $H_c \simeq 10^{12}$ T which is far above the maximal observed fields on the surface of neutron stars, but comparable with the field strength of the flux tubes in the proton condensate. The kaons are essentially non–relativistic so we can use Schafroth's [3] formula for the penetration depth $d = (m_K/4\pi e\rho_K)^{1/2} \simeq 3$ fm. Let us estimate the order of magnitude of the field gradient in the flux tube by H_c/d. Such a field gradient exerts an enormous force even on small magnetic dipoles. For instance, we estimate the force on the electron and the neutron, due to their intrinsic magnetic moments μ_e and μ_n, to be $\mathcal{O}(10^3)$ N and $\mathcal{O}(1)$ N, respectively.

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